Math 1433
Linear Algebra and Analytic Geometry

Monday 9 October 2023

Instructor: dr Adam Abrams

Topics

Vectors and matrices

- Vector operations
- Lines and planes
- Linear independence
- Matrix operations
- Systems of linear equations

Complex numbers and polynomials

- Rectangular and polar forms
- Complex conjugate
- Zeros of polynomials
- Factoring and remainders

Course website

All course policies can be found at

http://theadamabrams.com/1433

Lecture slides and problem sets will also be posted to this site throughout the semester.

Grades will be recorded on ePortal:

http://eportal.pwr.edu.pl/

Grading policy

The same grade is used for 1433W and 1433C.

- Six quizzes (5 points each), but the lowest score is ignored!
- Two exams (15 points each).
- Participation (5 points).

This makes 5×5 + 15+15 + 5 = **60** total possible points.

Points	[0, 30)	[30, 36)	[36, 42)	[42, 48)	[48, 54)	[54, 60]
Grade	2.0	3.0	3.5	4.0	4.5	5.0

This might change. If so, there will be an announcement.

Grading policy

The same grade is used for 1433W and 1433C.

Points	[0, 30)	[30, 36)	[36, 42)	[42, 48)	[48, 54)	[54, 60]
Grade	2.0	3.0	3.5	4.0	4.5	5.0

More than 3 unexcused absences after 13 Oct → course grade 2.0.

You can work together on task lists (which are not graded), but quizzes and exams are individual. *All work can be checked in one-on-one meeting.*

- Cheating on quizzes → quiz grade 0.
- Cheating on exams → course grade 2.0.

Accessibility

Accessibility and Support Department for People with Disabilities

- Office: building C-13 room 109
- Website: https://ddo.pwr.edu.pl/
- Email: pomoc.n@pwr.edu.pl

If you need extra time on exams, course materials in a different format, or other **accommodations**, please talk to me!

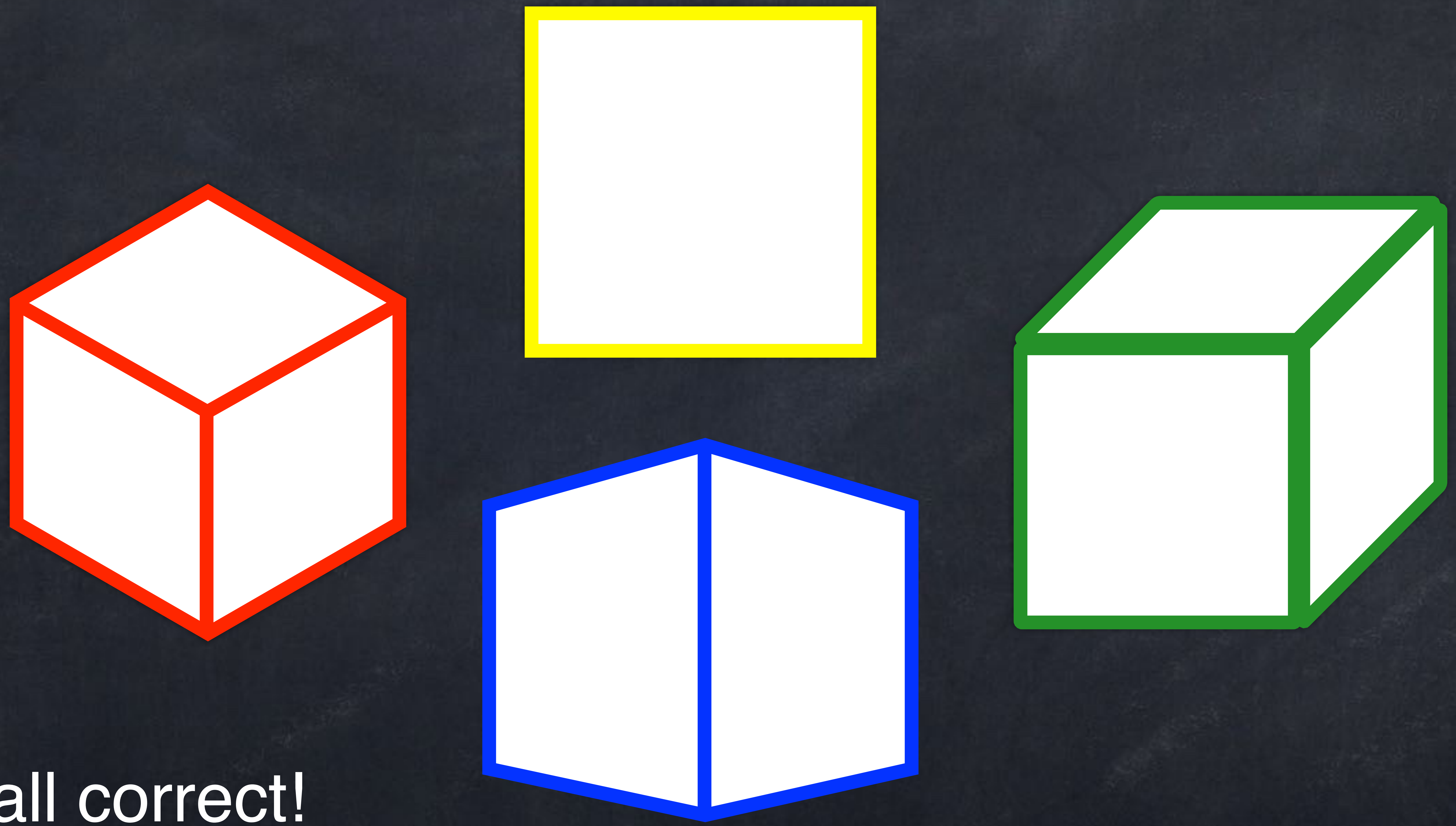
English language and some polls

poles

Poles

polls

Draw a cube



- These are all correct!
- If multiple people draw or talk about a cube, we need to be sure we are all thinking of the same thing.

Vectors as …

The word “vector” can mean many things.
At times we will think of a **vector** as

- a list.
- a point (that is, a location in 2D plane or 3D space).
- an arrow starting at the origin.
- an arrow starting anywhere.

There is another option:

- an element of an abstract vector space,

but we won’t use that idea of a vector in this class.

Vectors as Lists

Points
Arrows

A **vector** is a list of numbers.

- We can write the same list of numbers in many formats. For example,

$$(5,3,8) \qquad \langle 5,3,8\rangle \qquad [\,5\ \ 3\ \ 8\,] \qquad \begin{bmatrix}5\\3\\8\end{bmatrix} \qquad \begin{pmatrix}5\\3\\8\end{pmatrix}$$

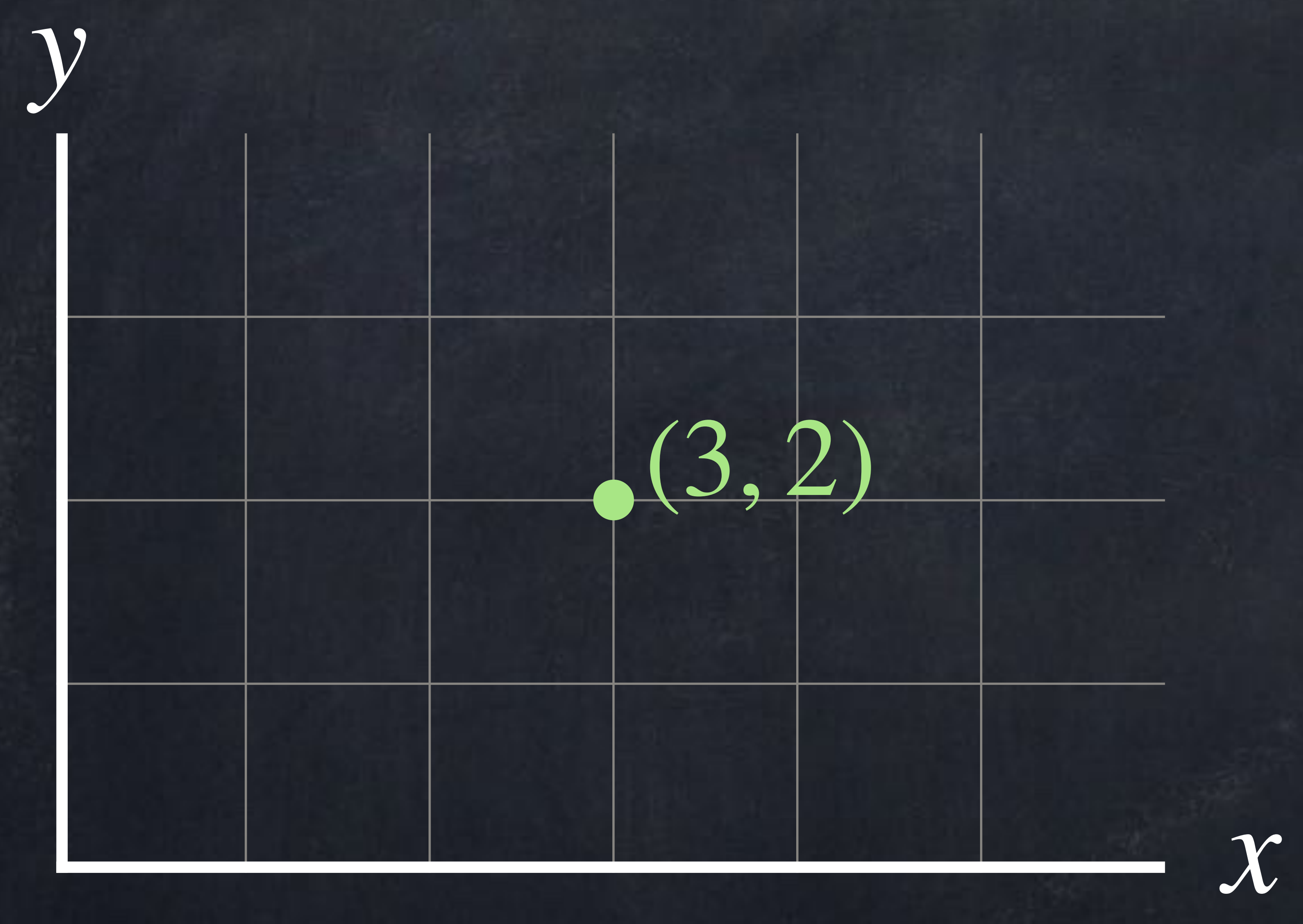
  are all exactly the same vector.
- Each numbers is a **component** of the vector.
  - For $[5,3,8]$, the "1st component" is 5, the "2nd component" is 3, etc.
- We often label the components with subscripts: $\vec{u} = \langle u_1, u_2, u_3\rangle$.

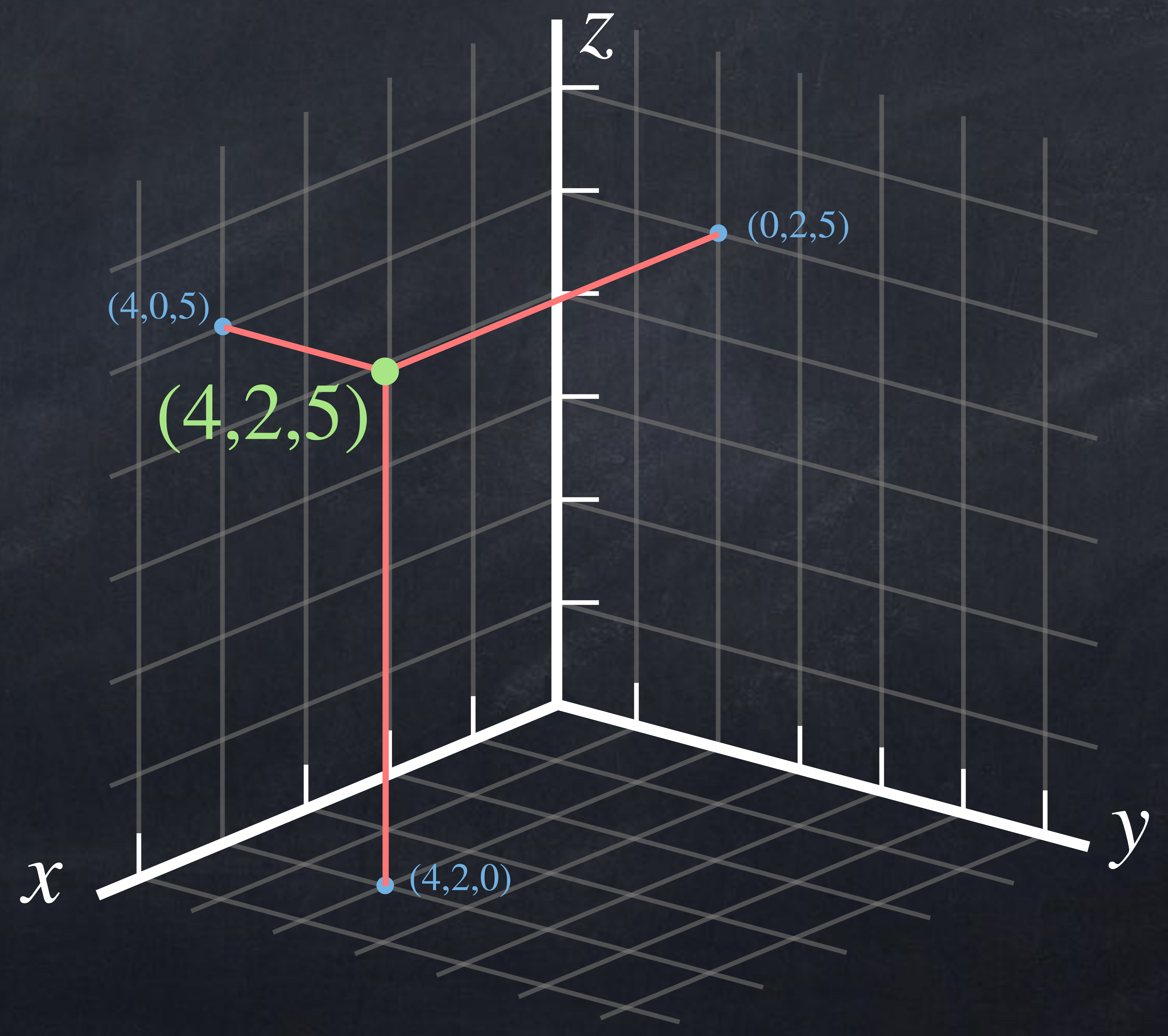
  But sometimes we instead label a whole vector this way: $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$.

Vectors as Points

Lists
Arrows

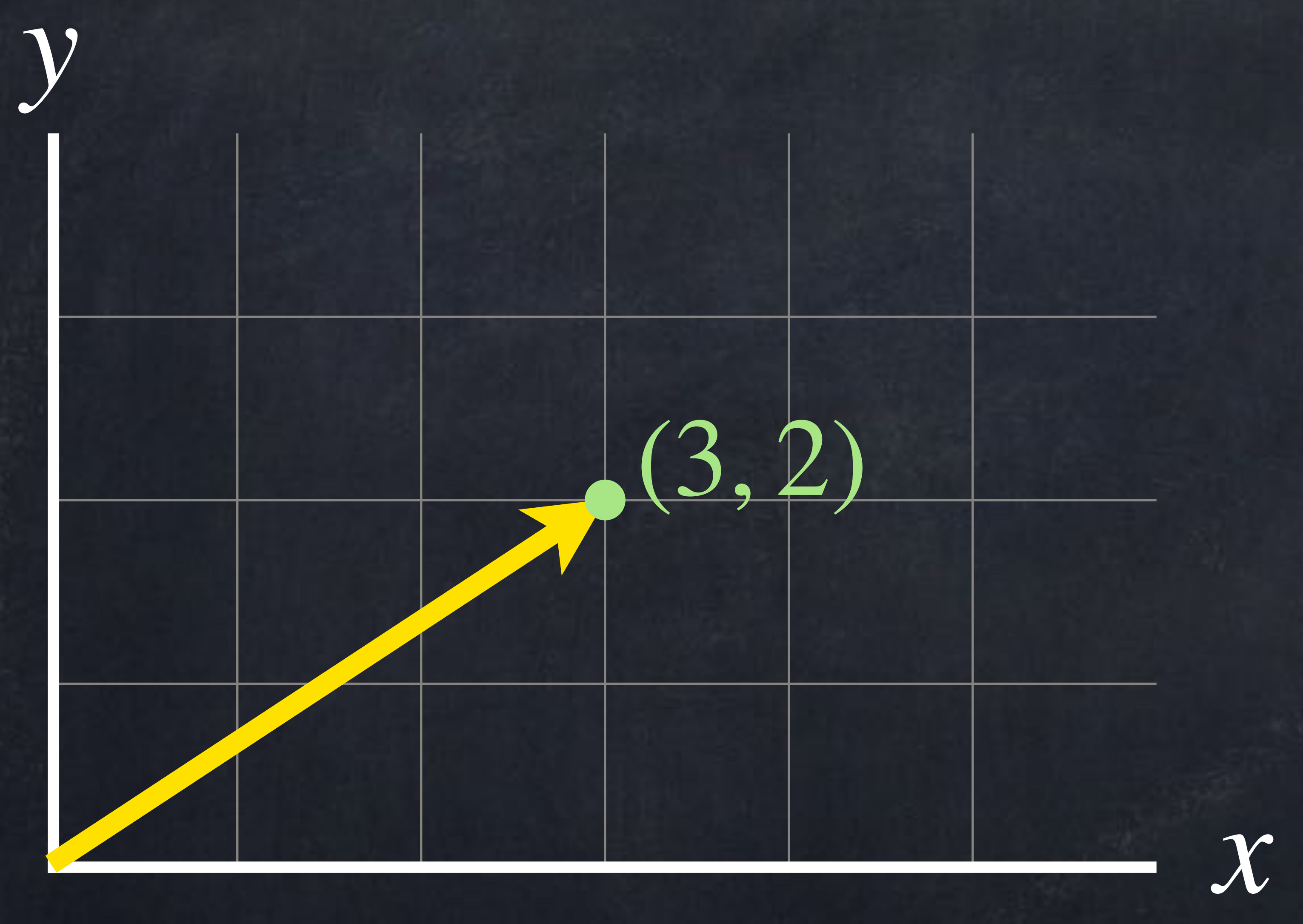
A **vector** is a point in 2D or 3D space.

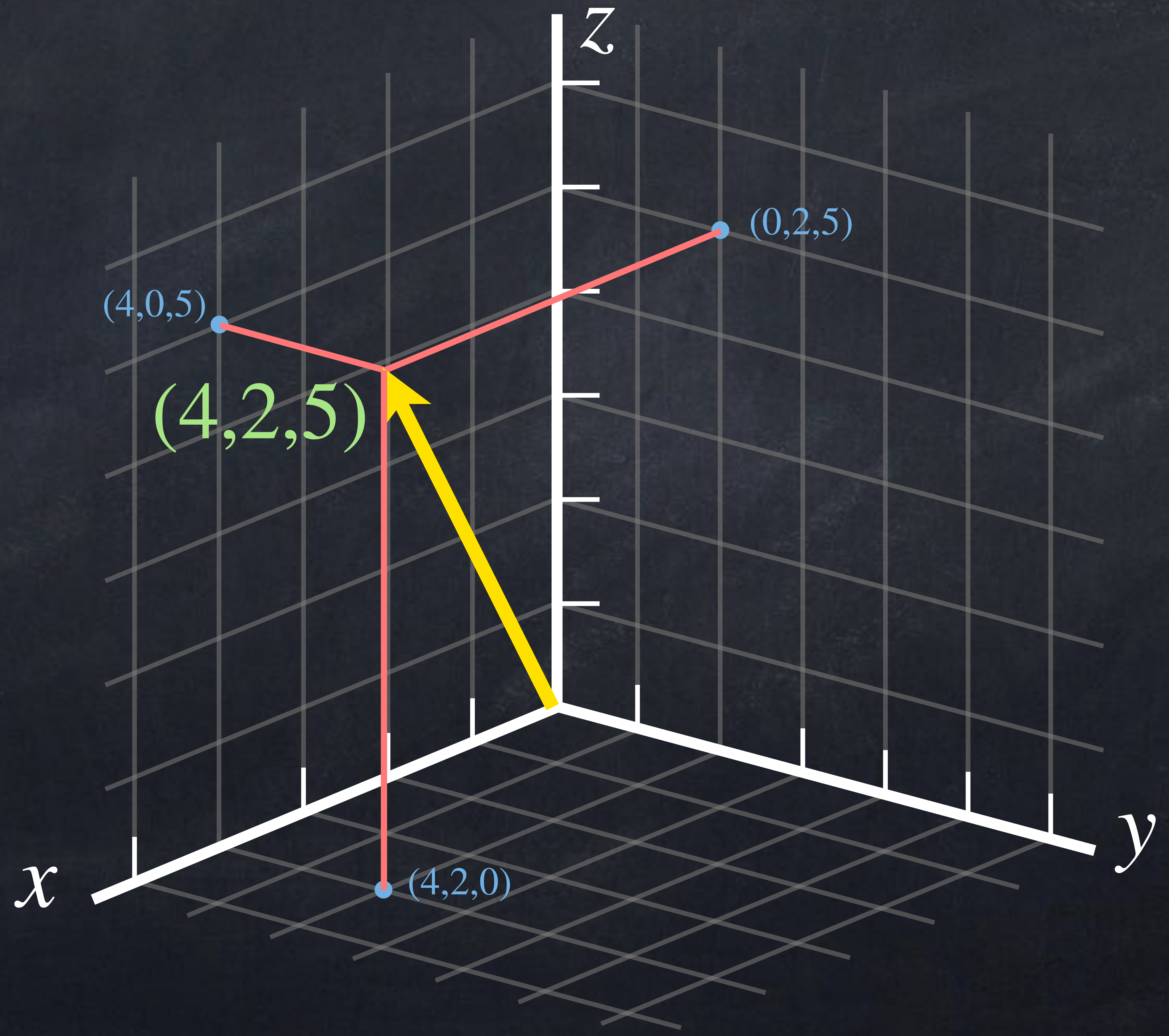




Vectors as ~~Lists~~ ~~Points~~ Arrows

A **vector** is something that has a magnitude and a direction.
In other words, it is an arrow.





Lists
Points

Vectors as Arrows

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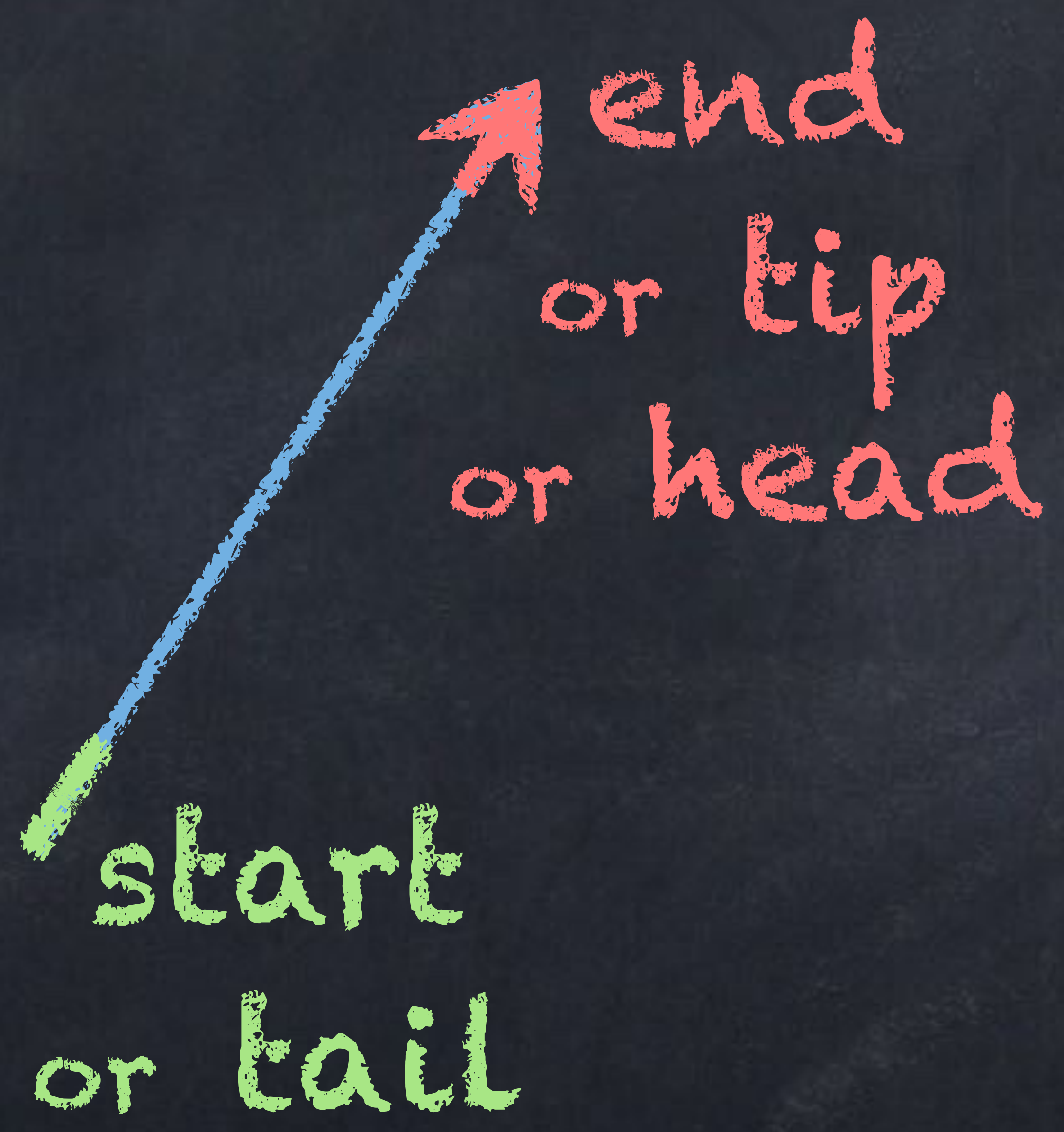

Vectors as ~~Lists~~ ~~Points~~ Arrows

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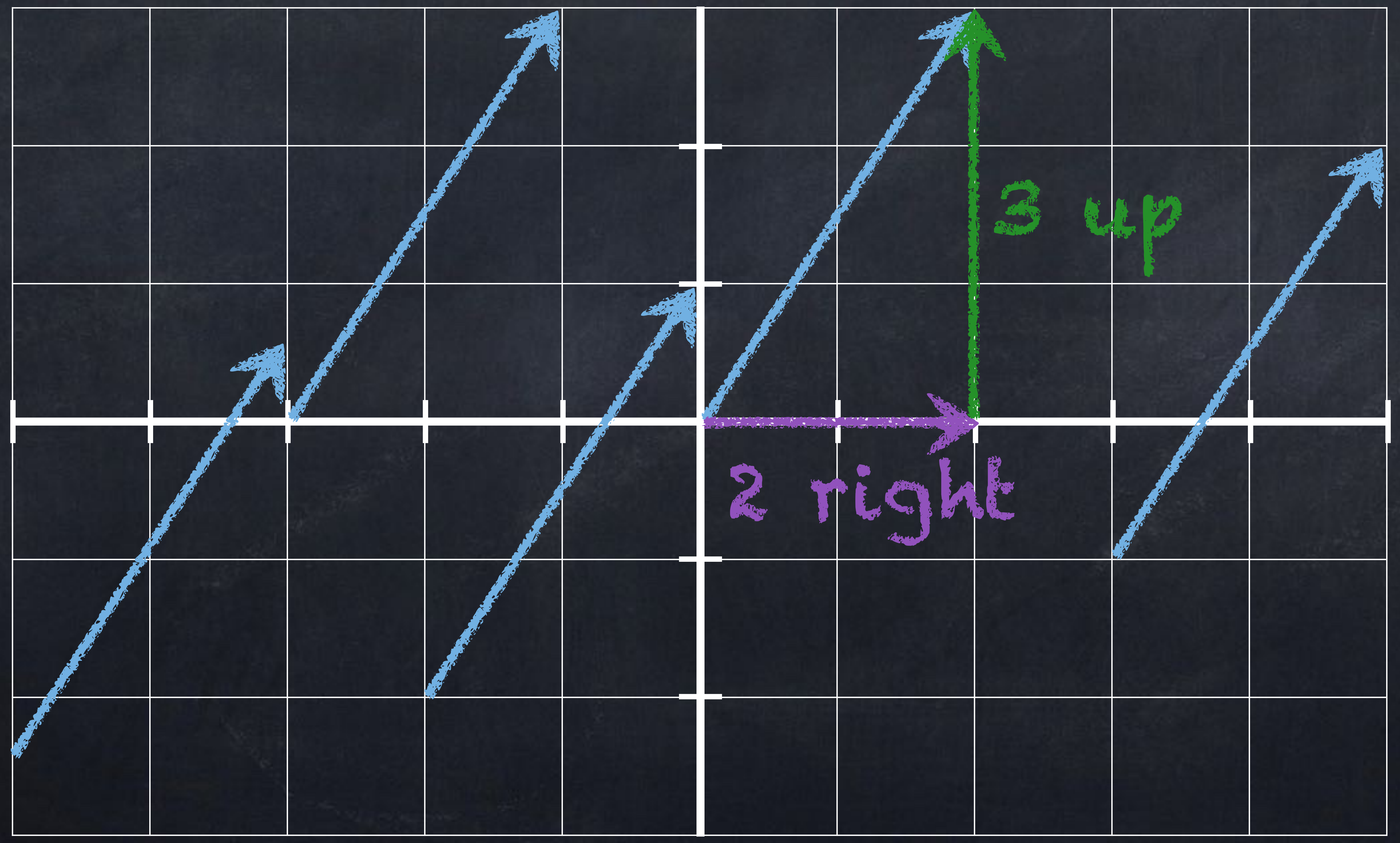
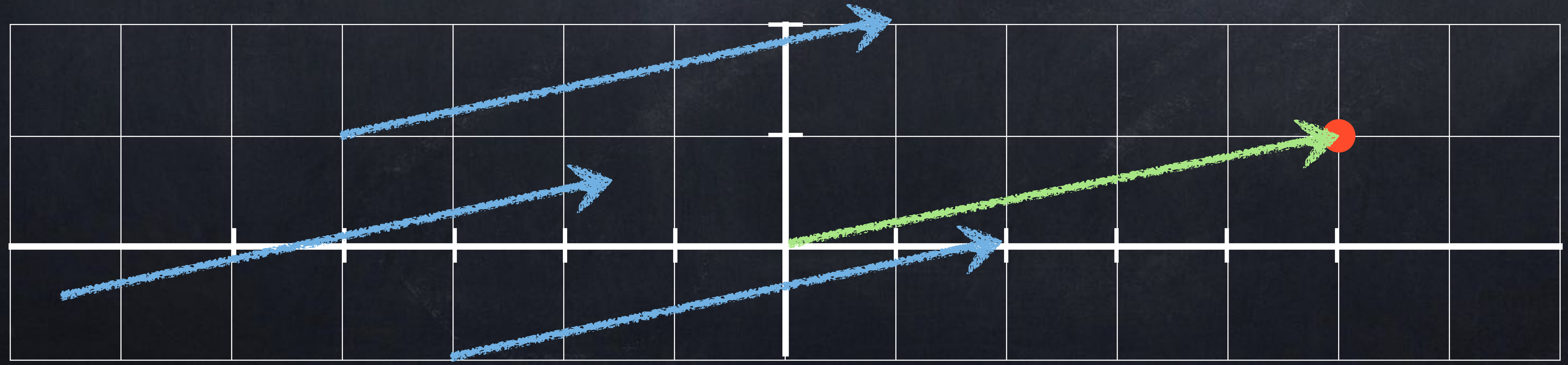


Vectors as Lists / Points / Arrows

A **vector** is something that has a magnitude and a direction.
In other words, it is an arrow.

Depending on context, a vector like $\langle 5,1 \rangle$ might refer to

- any arrow that points in a direction 5 right and 1 up, or
- the specific arrow from $(0,0)$ to $(5,1)$, or
- the point $(5,1)$.



Vector variables

In different text/videos, a vector variable might be written as any of these:

$$\vec{u} \qquad \overrightarrow{u} \qquad \boldsymbol{u} \qquad \mathbf{u} \qquad u$$

Often we use letters u, v, w or a, b, c for vectors.

If the vector has a specific meaning, we might use a letter related to that meaning.

- In physics, $\overrightarrow{F}$ for force.
- The **zero vector** is $\vec{0} = \langle 0,0 \rangle$ in 2D and $\vec{0} = \langle 0,0,0 \rangle$ in 3D.

Two vectors are **equal** if they have the same size and the components of the two vectors are equal (that is, their first components are equal, and their second components are equal, and so on).

- Example: $\langle 5, 1, 9 \rangle = \langle 2+3, \frac{6}{6}, 13-4 \rangle$

As with numbers, sometimes an equation describes one specific value

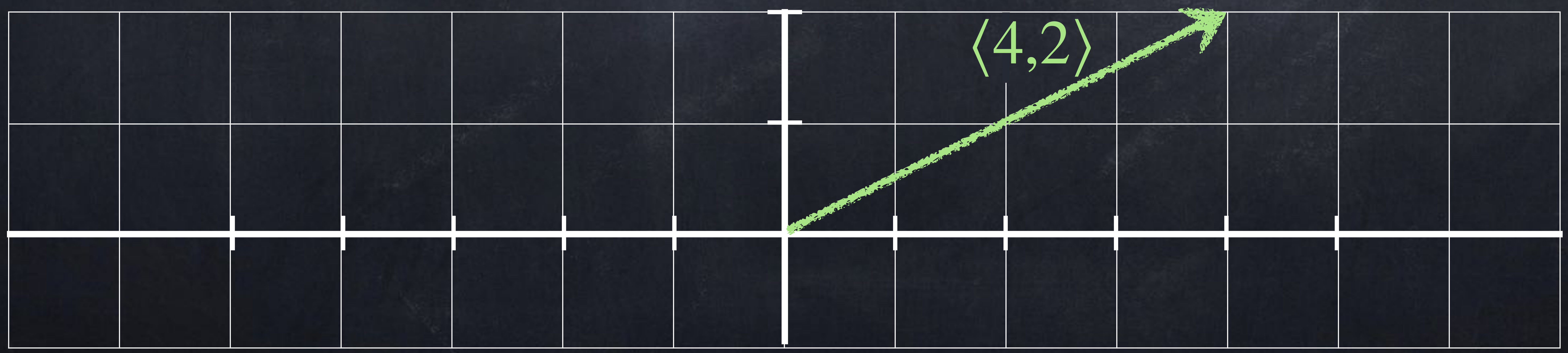
- $\vec{u} = \langle 1, -3 \rangle$
- $x = -8$
- and sometimes there are many values that make an equation true:
- $\vec{u} \;\; = 5$
- $x^2 - 4x + 3 = 0$

Magnitude

The **magnitude** (or **length** or **norm**) of the vector $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ is

$$\vec{v} \quad = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

In 2D or 3D, this is exactly the physical length of the arrow, or the length of the line segment from the origin to the point $\vec{v}$.



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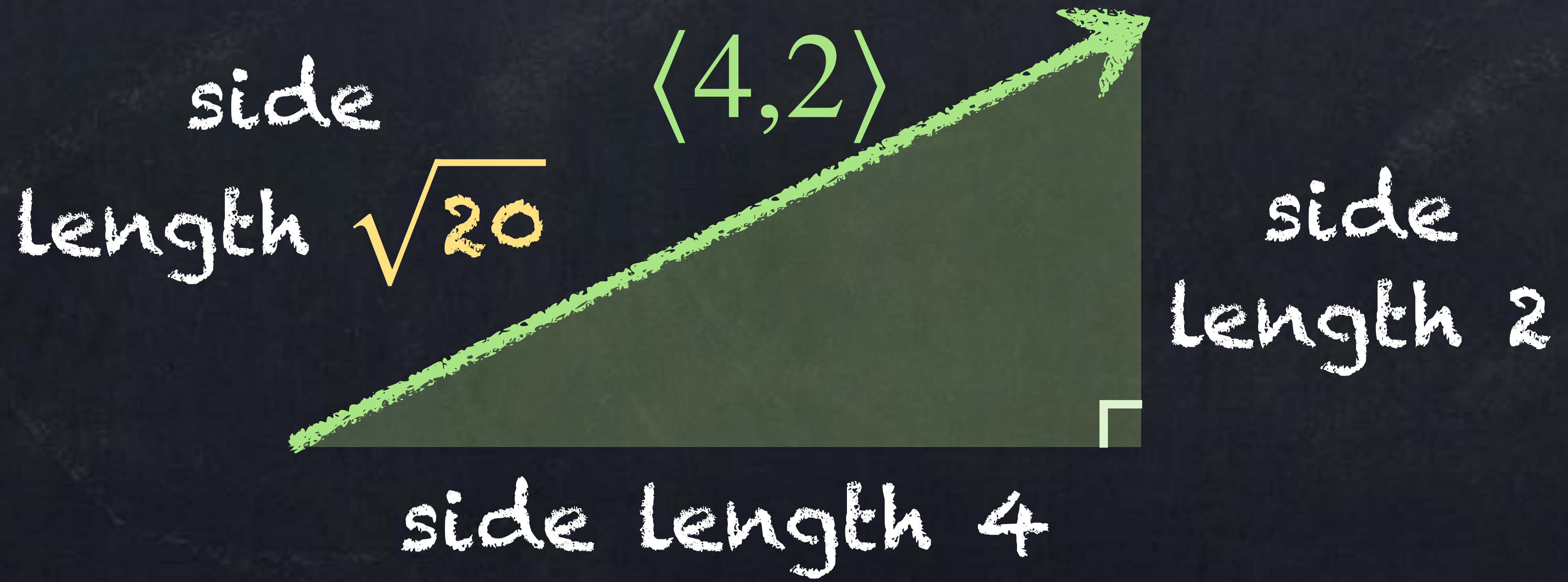
Example: for $\vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ we have $\vec{v} \quad = \sqrt{20} = 2\sqrt{5}$ because



Vector addition

As lists, vectors are added by adding each coordinate.

- Example: $\langle 9, -4\rangle + \langle 5, 6\rangle = \langle 14, 2\rangle$
- Example: $\begin{bmatrix} 5 \\ 8 \end{bmatrix} + \begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$

As arrows, add vectors "tip-to-tail" (also called "parallelogram method").

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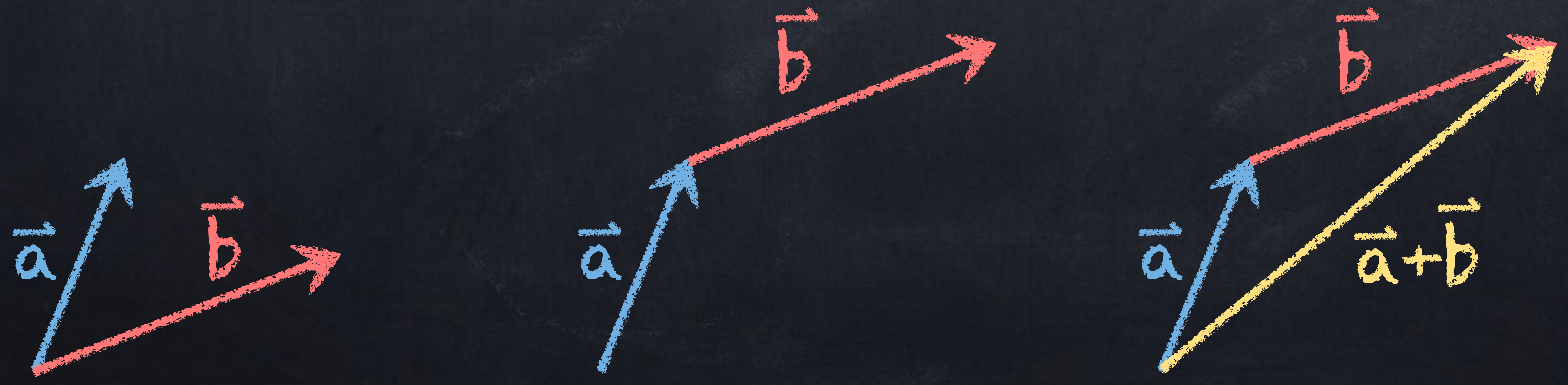

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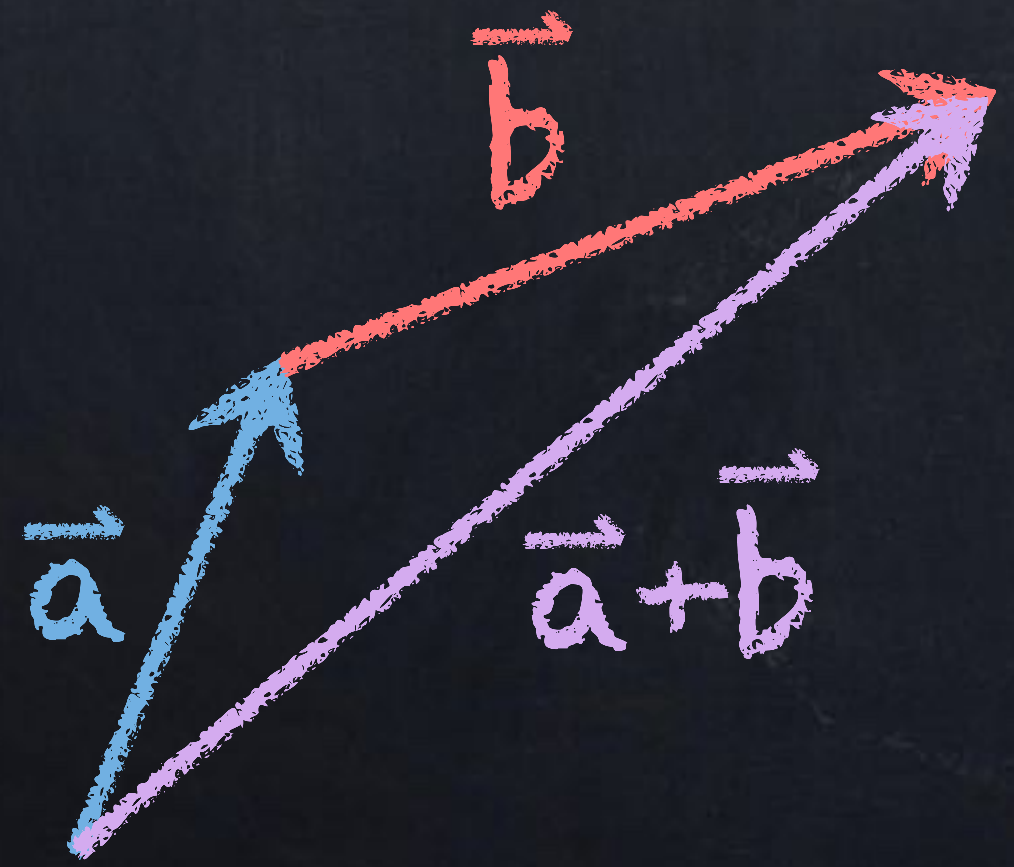
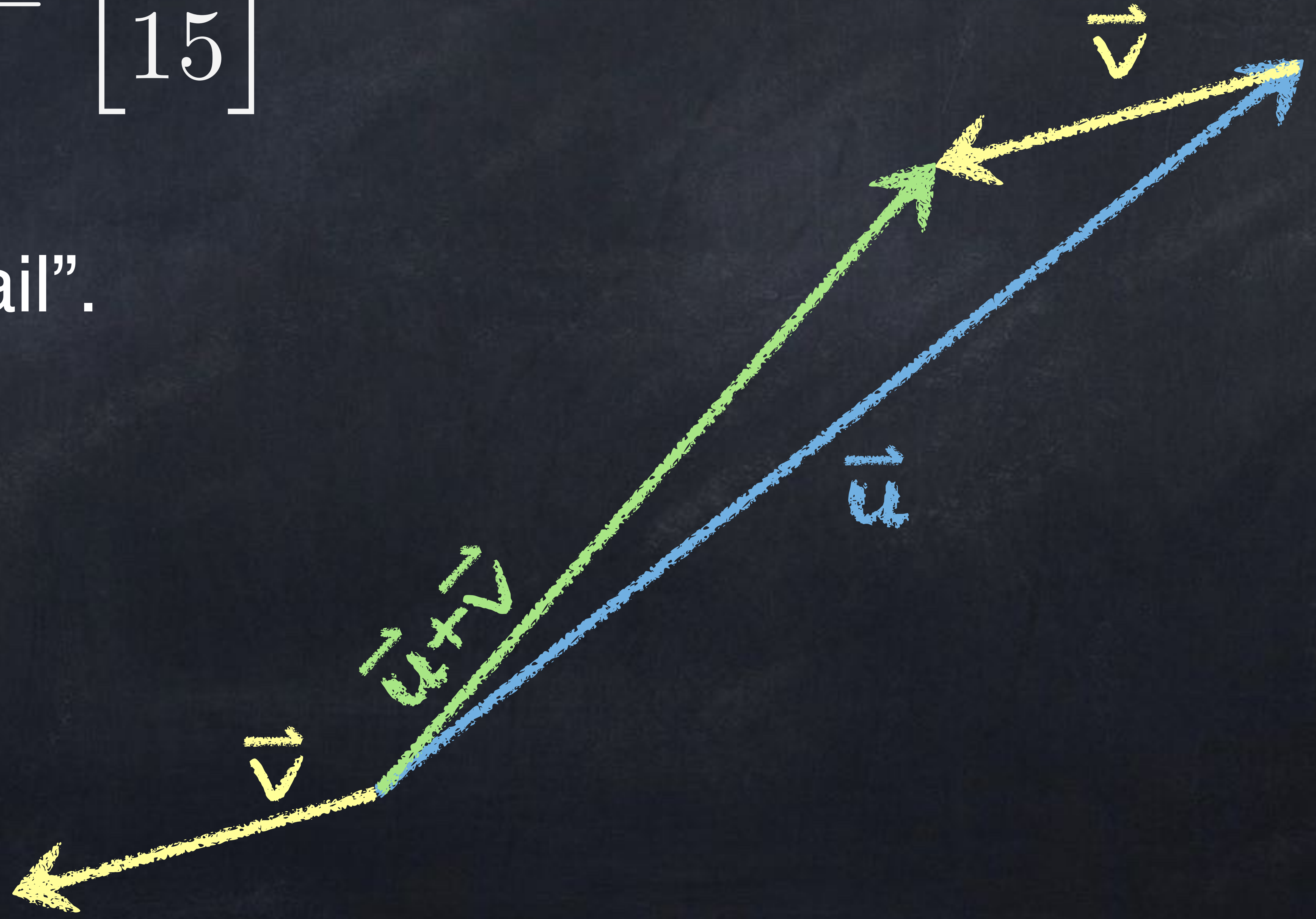




Multiplication

What does 5×3 mean?

- 
- More advanced: no pictures, just $5 + 5 + 5$.

What does $5 \times \frac{1}{3}$ mean? 5×9.2 ? $7.65 \times (-12)$?

- **Depending on the context, multiplication can have different meanings or interpretations.**
- This is also true for subtraction, and really for almost anything in math.

What is $\begin{bmatrix} 8 \\ -3 \end{bmatrix} + \begin{bmatrix} 8 \\ -3 \end{bmatrix}$?

We can also write this as $2\begin{bmatrix} 8 \\ -3 \end{bmatrix}$.

In general, $2\vec{a} = \vec{a} + \vec{a}$, and $3\vec{a} = \vec{a} + 2\vec{a}$, etc.

What about $2.5\,\vec{a}$ or $\sqrt{3}\vec{a}$?

Scalar multiplication

For this class, a **scalar** is a real number.

Given a scalar s and a vector $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$, we can multiply s and $\vec{v}$ to get

$$s\vec{v} = \langle sv_1, sv_2, \ldots, sv_n \rangle.$$

Examples:

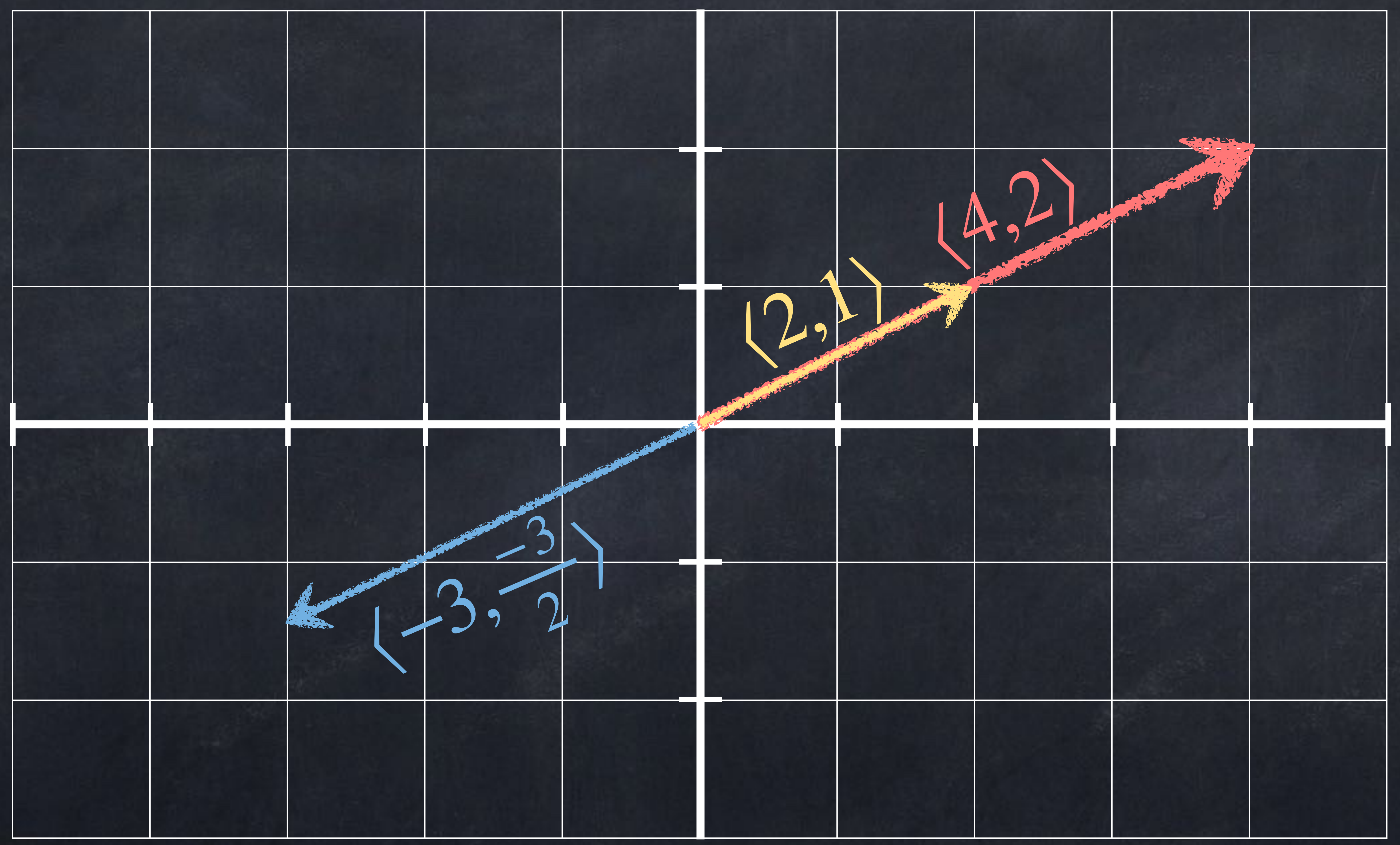
- $3\langle 8,1 \rangle = \langle 24,3 \rangle$
- $\frac{1}{2}\langle 8,1 \rangle = \langle 4,\frac{1}{2} \rangle$
- $4\langle -3, 9.1 \rangle = \langle -12, 36.8 \rangle$
- $-2\langle 5, -4 \rangle = \langle -10,8 \rangle$
- $0\langle 5,7 \rangle = \langle 0,0 \rangle$

We say that $\vec{a}$ is a **scalar multiple** of $\vec{b}$ if there is some number (scalar) s such that $\vec{a} = s\,\vec{b}$.

Examples:

- $\langle 24, 3\rangle$ is a scalar multiple of $\langle 8, 1\rangle$ (we can use $s = 3$).
- $\langle 4, 0.5\rangle$ is a scalar multiple of $\langle 8, 1\rangle$ (we can use $s = 0.5$).
- $\langle 24, 10\rangle$ is *not* a scalar multiple of $\langle 8,1\rangle$.

Geometrically, $s\vec{v}$ is a “stretched” version of $\vec{v}$.



Two vectors $\vec{u}$ and $\vec{v}$ are **parallel** if $\vec{u} = s\vec{v}$ for some $s \neq 0$.

- Some people require $s > 0$ and say, for example, that $\langle 3, 2 \rangle$ and $\langle -6, -4 \rangle$ are "anti-parallel". Some people call them parallel.
- $\vec{0}$ *is* a scalar multiple of any $\vec{v}$, but $\vec{0}$ is not parallel to $\vec{v}$.

Basis vectors

Later, we will talk about the general idea of a "basis", but for now we will use just one 2D example and one 3D example.

- In 2D, the **standard basis vectors** are

$$\vec{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It's also common to write "hats" instead of arrows above the letters: $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.

- In 3D, the **standard basis vectors** are

$$\vec{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We can write *any* vector using scalar multiples, these basis vectors, and vector addition.

Basis vectors

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Examples:

- $$\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}5\\0\end{bmatrix} + \begin{bmatrix}0\\2\end{bmatrix} = 5\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} = 5\vec{\imath} + 2\vec{\jmath}$$
- $$\begin{bmatrix}6\\0.91\\-2\end{bmatrix} = 6\vec{\imath} + 0.91\vec{\jmath} - 2\vec{k}$$
- $$\begin{bmatrix}a\\b\end{bmatrix} = a\vec{\imath} + b\vec{\jmath}$$
- $$\begin{bmatrix}4\\0\\1\end{bmatrix} = 4\vec{\imath} + \vec{k}$$
- $$\begin{bmatrix}5\\2\\0\end{bmatrix} = 5\vec{\imath} + 2\vec{\jmath}$$

Subtraction

We can subtract vectors using coordinates.

- Example: $\langle 9, -4 \rangle \ - \ \langle 5, 6 \rangle \ = \ \langle 4, \ -10 \rangle$
- Example: $$\begin{bmatrix} 5 \\ 8 \end{bmatrix} - \begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

What does $\vec{u} - \vec{v}$ mean geometrically?

- We *could* first find the scalar multiple $-\vec{v} = (-1)\vec{v}$ and then use the geometric method of addition ("tip-to-tail") to draw $\vec{u} + (-\vec{v})$.
- What does $a - b$ mean for numbers?

Subtraction

What does $5 - 3$ mean **on a number line**?

Answer: The number $5 - 3$ describes how to move from 3 to 5.

In general, $a - b$ describes how to move from b to a.

Subtraction

What does $5 - 3$ mean **on a number line**?

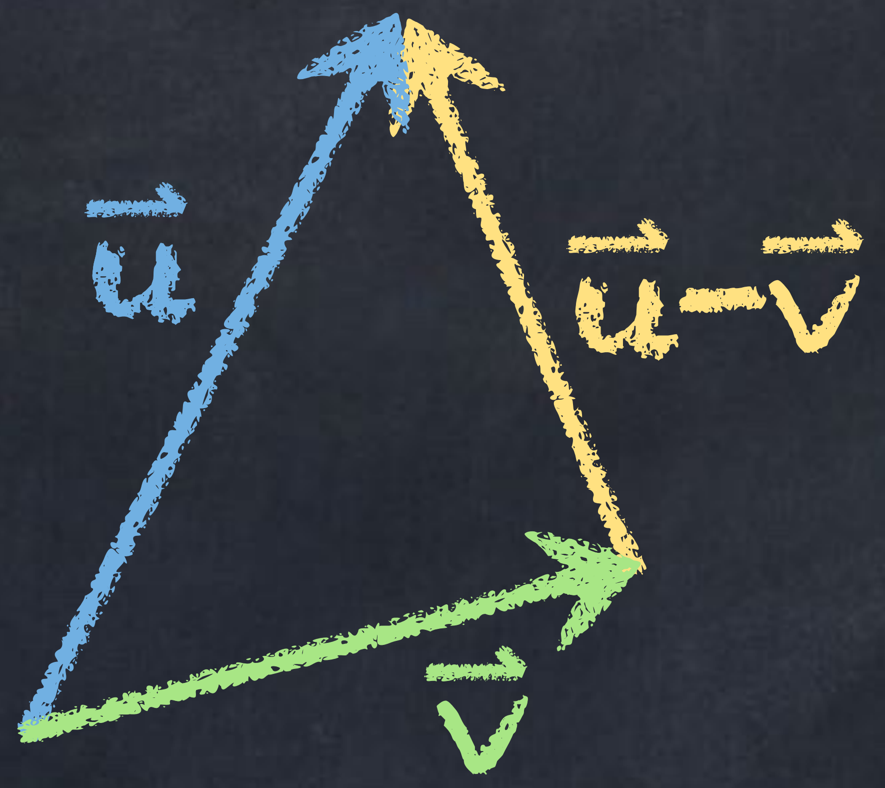
Answer: The number $5 - 3$ describes how to move from 3 to 5.

- To go from 5 to 3 instead, we move *left*, which is why $3 - 5$ is negative.

In general, $a - b$ describes how to move from b to a.

Subtraction

The vector $\vec{u} - \vec{v}$ points from the end of $\vec{v}$ to the end of $\vec{u}$.



Note: The tails (start) of $\vec{u}$ and $\vec{v}$ must be at the same place to use this method.

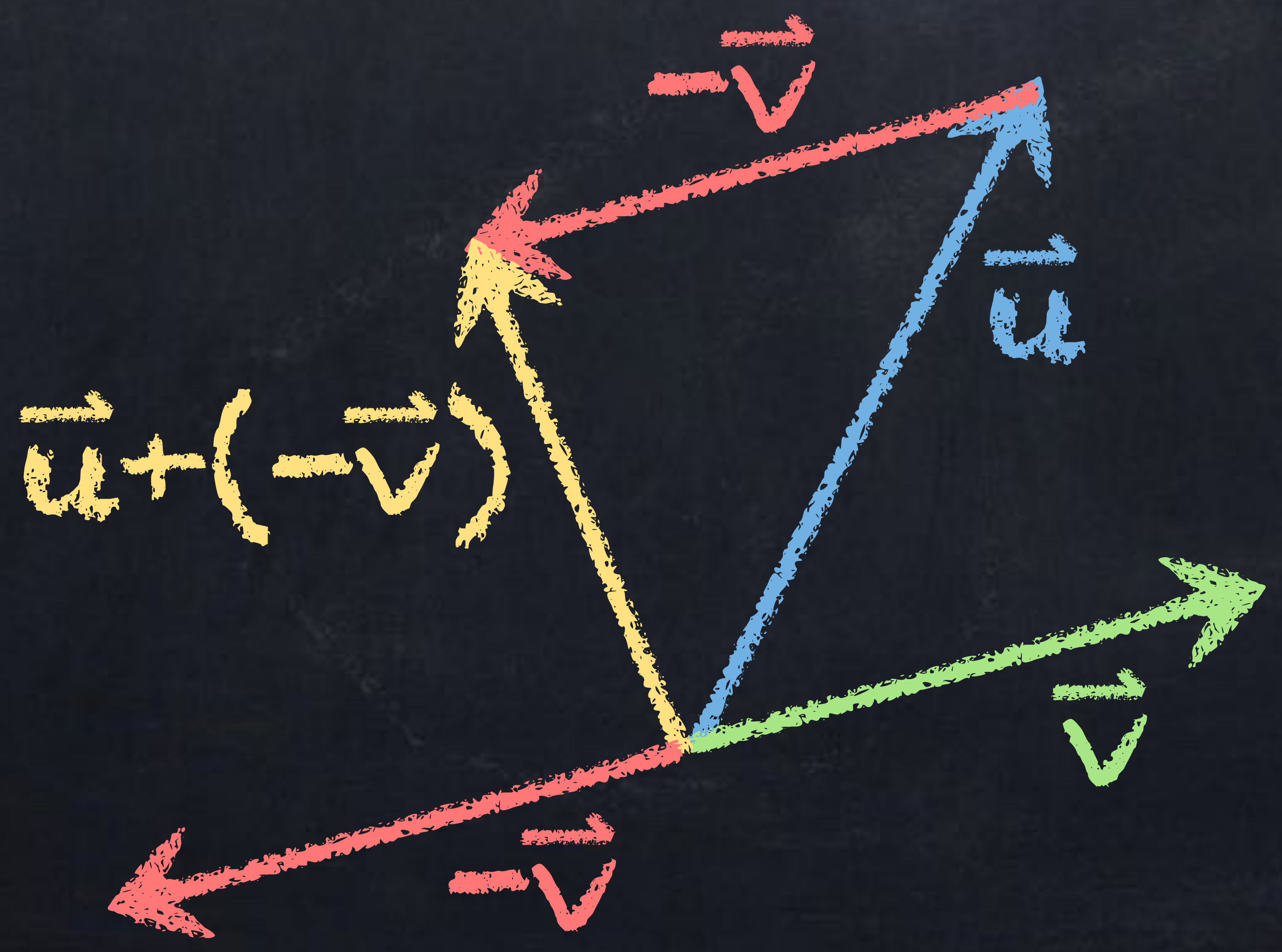
In general, $a - b$ describes how to move from b to a.

Subtraction

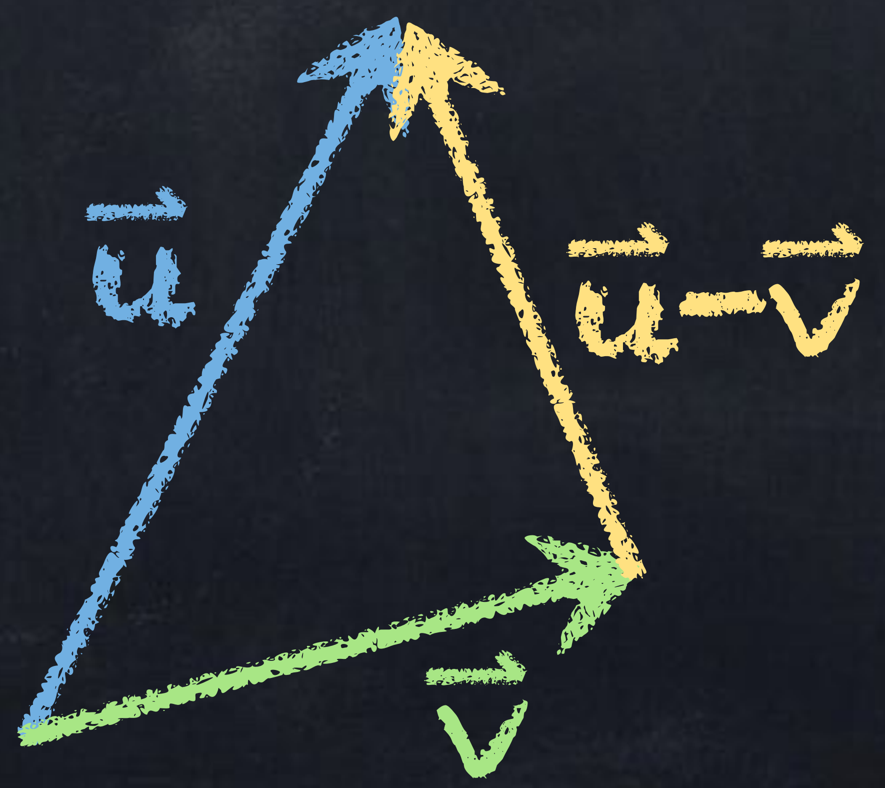
The vector $\vec{u} - \vec{v}$ points from the end of $\vec{v}$ to the end of $\vec{u}$.

Note: The tails (start) of $\vec{u}$ and $\vec{v}$ must be at the same place to use this method.

- This agrees with finding $\vec{u} - \vec{v}$ by adding $\vec{u} + (-\vec{v})$ tip-to-tail.

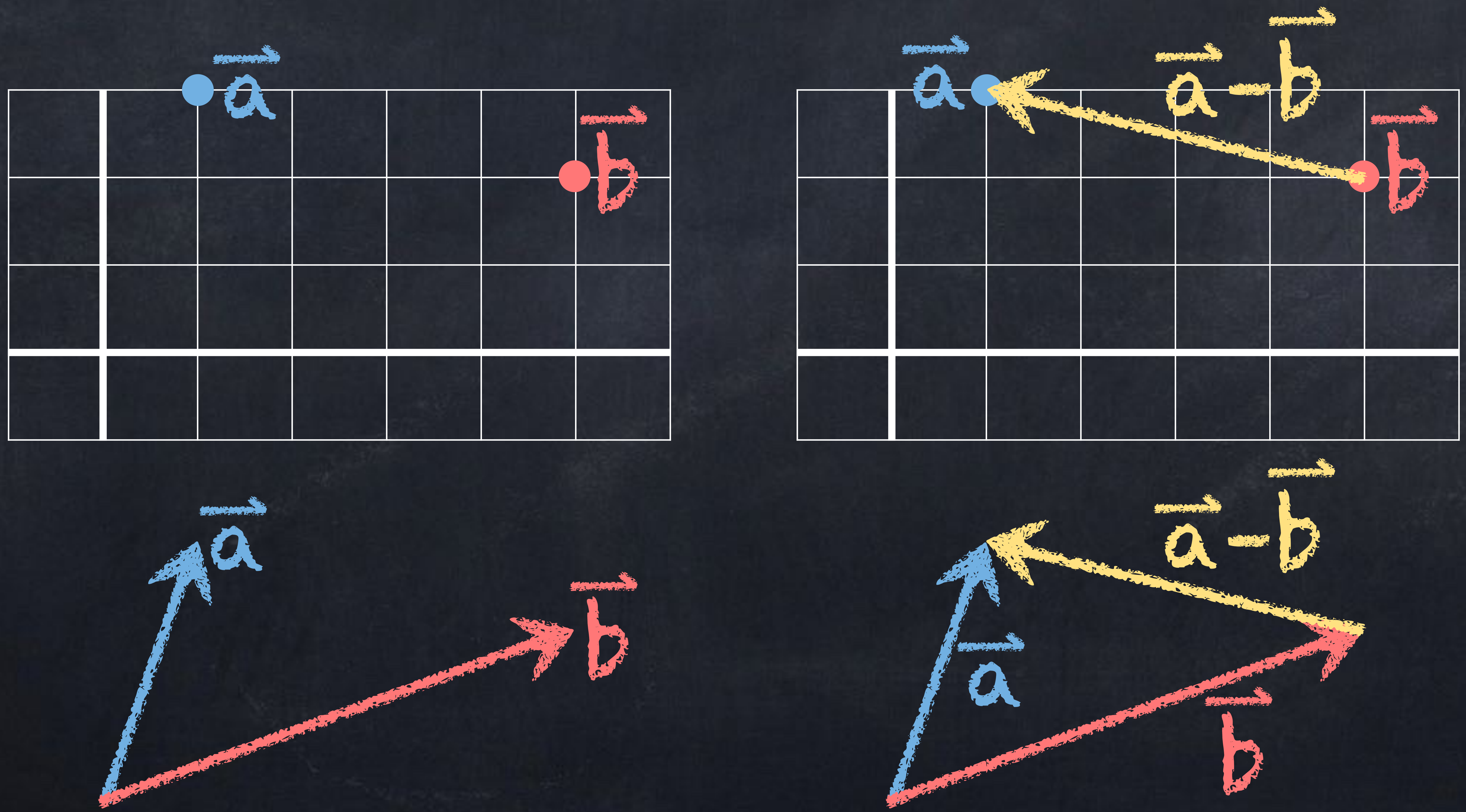


Vector $\vec{u} - \vec{v}$ points from the end of $\vec{v}$ to the end of $\vec{u}$.



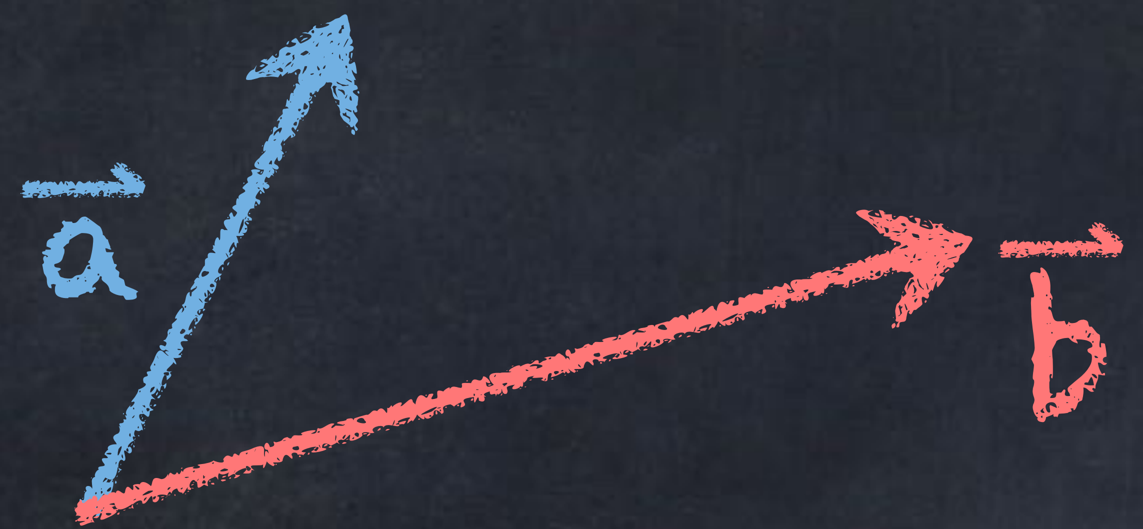
Note: The tails (start) of $\vec{u}$ and $\vec{v}$ must be at the same place to use this method.

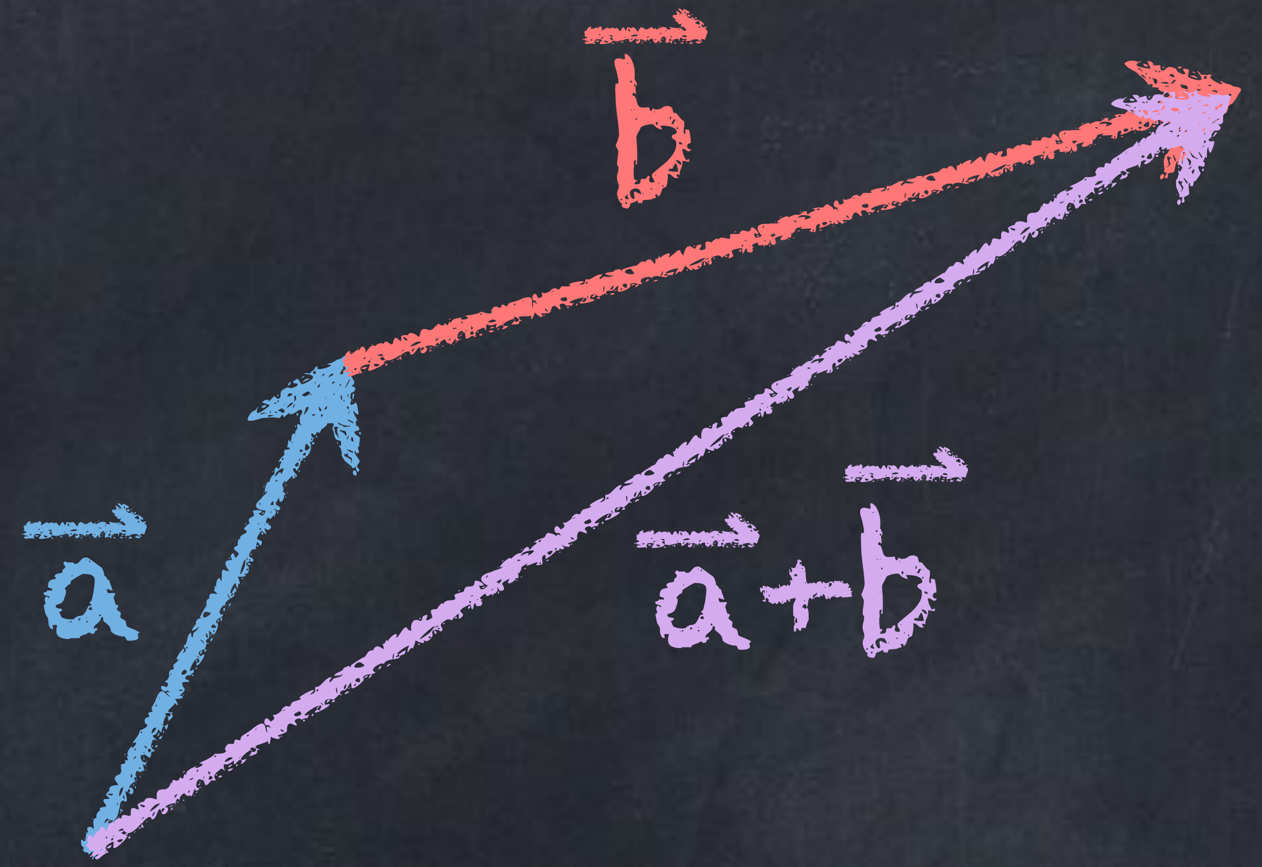
If we think of $\vec{a}$ and $\vec{b}$ as ***points***, then $\vec{a} - \vec{b}$ literally goes from $\vec{b}$ to $\vec{a}$.



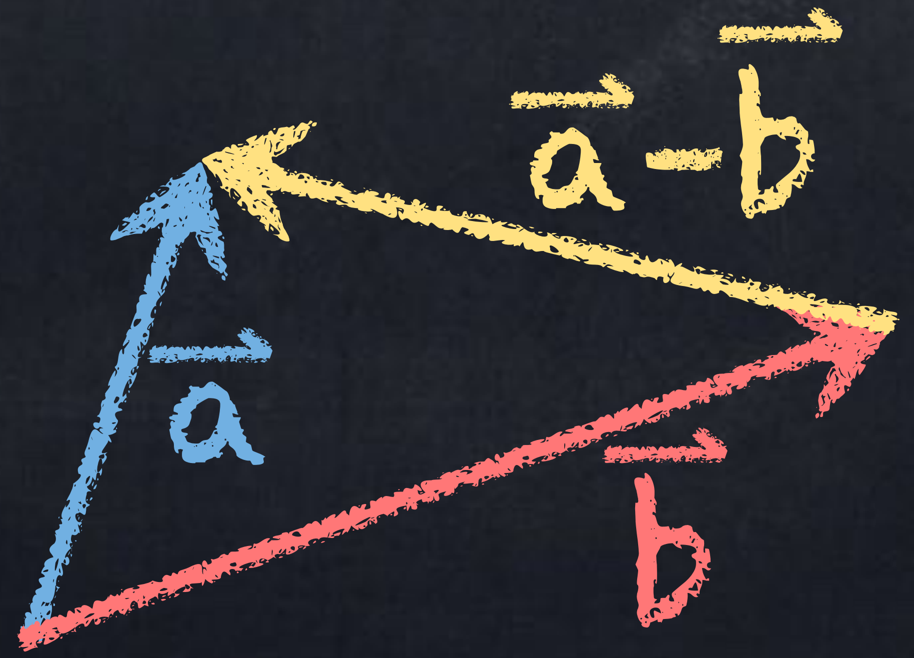
Summary: + and −

- In terms of arrows, we have "tip-to-tail addition". Example:





- If $\vec{a}, \vec{b}$ start at the same point, then $\vec{a} - \vec{b}$ points from the end of $\vec{b}$ to the end of $\vec{a}$.



Multiplication

There are actually many kinds of multiplication involving vectors:

- Scalar multiple $s\,\vec{u}$
- Dot product $\vec{u} \cdot \vec{v}$
- Cross product $\vec{u} \times \vec{v}$
- Outer product $\vec{u}^\top \vec{v}$ for rows
- Convolution $\vec{u} * \vec{v}$
- Kronecker product $\vec{u} \otimes \vec{v}$
- Hadamard product $\vec{u} \odot \vec{v}$

In our class, we'll only use these.

Never write $\vec{a}\vec{b}$ without a symbol in between the vectors!

For fractions, $\frac{1}{2} \cdot \frac{1}{3} = \frac{1 \cdot 1}{2 \cdot 3}$ (easy), but $\frac{1}{2} + \frac{1}{3}$ is *not* $\frac{1+1}{2+3}$.

For vectors, it's the opposite:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 2+3 \end{bmatrix}$$ (easy).

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is *not* $\begin{bmatrix} 1 \cdot 1 \\ 2 \cdot 3 \end{bmatrix}$. If you do this calculation on a quiz or exam, you will lose points.

Dot product / scalar product

The **dot product** of two vectors $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$,

also called the **scalar product** or **inner product**, is written as $\vec{a} \cdot \vec{b}$ (said out loud as “A dot B”). It is a *number* that can be computed as either

- $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

or

- $\vec{a} \cdot \vec{b} = \left|\vec{a}\right| \left|\vec{b}\right| \cos\left(\text{angle between } \vec{a} \text{ and } \vec{b}\right)$.

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Today's schedule

Officially there is a problem session from 12:15 - 13:00 today.

- We will discuss the topics mentioned on the survey.
- If you can already answer the tasks below, you can skip today's session.

Algebra	Expand $(2x+3)(x-5)$.
Lines	Does the point $(3,8)$ lie on the line $y=4+8(x-3)$? The point $(3,-20)$? The point $(3,4)$?
Quadratic f.	Solve $x^2-2x-9=0$.
Exponents	Re-write $(2^3)^2 \cdot 2^{10}$ in the form $2^{\square}$.
Systems	Solve $\begin{cases} 3x+1=4 \\ x-y=3. \end{cases}$
Trig	What is $\cos(30°)$? Find a value of θ for which $\sin(\theta)=\frac{1}{\sqrt{2}}$ *and* $\cos(\theta)=\frac{-1}{\sqrt{2}}$.